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Time evolution of a superposition of dressed oscillator states in a cavity

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Abstract

Using the formalism of *renormalized* coordinates and *dressed* states introduced in previous publications, we perform a nonperturbative study of the time evolution of a superposition of two states, the ground state and the first excited level of a harmonic oscillator, the system being confined in a perfectly reflecting cavity of radius R. For $R \to \infty$, we find dissipation with dominance of the interference terms of the density matrix, in both weak- and strong-coupling regimes. For small values of R all elements of the density matrix present an oscillatory behavior as time goes on and the system is not dissipative. In both cases, we obtain improved theoretical results with respect to those coming from perturbation theory.

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1. Introduction

The main analytical method used to treat the physics of interacting systems is perturbation theory. In this framework, *bare*, noninteracting, matter fields and gauge fields (with which bare quanta are associated) are introduced, the interaction being accounted for order by order in powers of the coupling constant in the perturbative expansion of the observables. For a long time, physicists have been aware that perturbative expansions have a limited scope, in spite of all the remarkable achievements obtained with them. Some examples of such situations are resonant effects associated with the coupling of atoms to strong radiofrequency fields, or the low-energy domain of quantum chromodynamics, where confinement of quarks and gluons takes place. Along the last few decades, many attempts were devised to circumvent the limitations of perturbation theory, in particular when strong effective couplings are involved.

In any case, as a matter of principle, due to the nonvanishing of the coupling constant, the idea of a bare particle associated with a bare matter field is actually an artifact of perturbation theory and strictly speaking is physically meaningless. In general, a charged physical particle is always coupled to a gauge field; in other words, it is always 'dressed' by a cloud of quanta of the gauge field, for instance, photons in the case of electrodynamics.

With respect to the time evolution of systems of (matter) particles, the idea is that particles are coupled to an environment or to a thermal bath. There are usually two equivalent ways of modeling the environment to which the particle is coupled: either to represent it by a free field, as was done in [1, 2], or to consider the environment as a reservoir composed of a large number of noninteracting harmonic oscillators (see, for instance, [3–6]). In both cases, exactly the same type of argument given above for a charged particle applies, making the appropriate changes, to these systems. We may speak of a 'dressing' of the set of particles by the ensemble of the harmonic modes of the environment. The particles in the system are considered as always 'dressed' by a cloud of quanta of the environment. This is true, in general, for any system in which material particles are coupled to an environment, no matter what the specific nature of the environment and interaction involved is.

In the present work we adopt this point of view and consider a harmonic oscillator, coupled linearly to an environment modeled by the infinite set of harmonic modes of a scalar field, the whole system being contained in a perfectly reflecting sphere of radius R. We implement a nonperturbative study of the time evolution of a superposition of the first excited and the ground states of the oscillator, by means of the *dressed* states introduced in [7] and already employed to investigate several situations [8–13]. Such a nonperturbative approach is possible due to the linear character of the interaction between the oscillator and the environment we choose.

The semiqualitative idea of a 'dressed atom' was originally introduced in [14, 15], and largely employed in studies involving the interaction of atoms and electromagnetic fields [18–21]. In the realm of general physics, the concept of dressing a matter particle by an environment has found an application in describing the radiation damping of classical systems [16]. Our dressed states can be viewed as a rigorous version of these dressing procedures. Moreover, the oscillator may represent a mode of the quantized electromagnetic field in a cavity interacting with the environment; in this case, our study refers to the time evolution of the superposition of the ground state and the first excited state of the field in the cavity.

The model in itself is not new; a system composed of a material body interacting linearly with an environment has been the main subject of many papers, as those quoted in [1–6, 16, 17]. The novelty lies in the nonperturbative approach to the problem, by means of *renormalized* coordinates and *dressed* states, which was started in [7]. In particular, our dressed states are *not* the same as those employed in the literature, usually associated with normal coordinates. Our dressed states are given in terms of our *renormalized* coordinates and allow a rigorous study of the time evolution of quantum systems. The results we obtain by this means are those expected on physical grounds, but contain corrections with respect to the formulae obtained from perturbation theory.

We structure the paper as follows. In section 2, we review the basic aspects of the formalism of renormalized coordinates and dressed states and then, in section 3, we treat the time evolution of a superposition of oscillator states in both limiting situations of $R \to \infty$ and small R. Section 4 contains our final comments.

2. Renormalized coordinates and dressed states

Let us start by considering the harmonic oscillator having *bare* frequency ω_0, linearly coupled to a field described by N ($\rightarrow\infty$) other oscillators, with frequencies ω_k, $k = 1, 2, \ldots, N$. The whole system is contained in a perfectly reflecting spherical cavity of radius R, the free space corresponding to the limit $R \to \infty$. Hereafter, we shall refer to the harmonic oscillator as the *particle*, to distinguish it from the harmonic modes of the environment. Denoting by $q_0(t)$ ($p_0(t)$) and $q_k(t)$ ($p_k(t)$) the coordinates (momenta) associated with the particle and the field oscillators respectively, the Hamiltonian of the system is

$$H = \frac{1}{2}\left[p_0^2 + \omega_0^2 q_0^2 + \sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2\right)\right] - q_0\sum_{k=1}^{N}\eta\omega_k q_k, \tag{1}$$

where the limit $N \to \infty$ is understood and η is a constant.

Hamiltonian (1) can be turned to a principal axis by means of a point transformation,

$$q_\mu = \sum_{r=0}^{N} t_\mu^r Q_r, \qquad p_\mu = \sum_{r=0}^{N} t_\mu^r P_r, \tag{2}$$

where $\mu = (0, \{k\}), k = 1, 2, \ldots, N$, performed by an orthonormal matrix $T = \left(t_\mu^r\right)$. The subscripts $\mu = 0$ and $\mu = k$ refer respectively to the particle and the harmonic modes of the field and r refers to the normal modes. In terms of normal momenta and coordinates, the transformed Hamiltonian reads

$$H = \frac{1}{2}\sum_{r=0}^{N}\left(P_r^2 + \Omega_r^2 Q_r^2\right), \tag{3}$$

where the Ω_r's are the normal frequencies corresponding to the collective *stable* oscillation modes of the coupled system.

Using the coordinate transformation (2) in the equations of motion and explicitly making use of the normalization condition

$$\sum_{\mu=0}^{N}\left(t_\mu^r\right)^2 = 1, \tag{4}$$

we get

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r, \qquad t_0^r = \left[1 + \sum_{k=1}^{N}\frac{\eta^2\omega_k^2}{\left(\omega_k^2 - \Omega_r^2\right)^2}\right]^{-\frac{1}{2}}, \tag{5}$$

with the condition

$$\omega_0^2 - \Omega_r^2 = \sum_{k=1}^{N}\frac{\eta^2\omega_k^2}{\omega_k^2 - \Omega_r^2}. \tag{6}$$

The right-hand side of equation (6) diverges in the limit $N \to \infty$. Defining the counterterm $\delta\omega^2 = N\eta^2$, it can be rewritten in the form

$$\omega_0^2 - \delta\omega^2 - \Omega_r^2 = \eta^2\Omega_r^2\sum_{k=1}^{N}\frac{1}{\omega_k^2 - \Omega_r^2}. \tag{7}$$

Equation (7) has $N + 1$ solutions, corresponding to the $N + 1$ normal collective modes. It can be shown [7] that if $\omega_0^2 > \delta\omega^2$, all possible solutions for Ω^2 are positive, physically meaning that the system oscillates harmonically in all its modes. On the other hand, when

$\omega_0^2 < \delta\omega^2$, one of the solutions is negative and so no stationary configuration is allowed. Nevertheless, in a different context, it is precisely this runaway solution that is related to the existence of a bound state in the Lee–Friedrichs model. This solution is considered in [22] in the framework of a model to describe qualitatively the existence of bound states in particle physics.

Therefore, we just consider the situation in which all normal modes are harmonic, which corresponds to the above first case, $\omega_0^2 > \delta\omega^2$, and define the *renormalized* frequency

$$\bar{\omega}^2 = \lim_{N\to\infty} \left(\omega_0^2 - N\eta^2\right), \tag{8}$$

following the pioneering work of [23]. In the limit $N \to \infty$, equation (7) becomes

$$\bar{\omega}^2 - \Omega^2 = \eta^2 \sum_{k=1}^{\infty} \frac{\Omega^2}{\omega_k^2 - \Omega^2}. \tag{9}$$

We see that, in this limit, the above procedure is exactly the analogue of mass renormalization in quantum field theory: the addition of a counterterm $-N\eta^2 q_0^2$ allows one to compensate the infinity of ω_0^2 in such a way as to leave a finite, physically meaningful, renormalized frequency $\bar{\omega}$.

To proceed with, we take the constant η as

$$\eta = \sqrt{\frac{4g\Delta\omega}{\pi}}, \tag{10}$$

where $\Delta\omega$ is the interval between two neighboring field frequencies and g is the coupling constant with dimension of frequency. The environment frequencies ω_k can be written in the form

$$\omega_k = k\frac{\pi c}{R}, \qquad k = 1, 2, \ldots, \tag{11}$$

and, so, $\Delta\omega = \pi c/R$. Then, using the identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - u^2} = \frac{1}{2}\left[\frac{1}{u^2} - \frac{\pi}{u}\cot(\pi u)\right], \tag{12}$$

equation (9) can be written in a closed form:

$$\cot\left(\frac{R\Omega}{c}\right) = \frac{\Omega}{2g} + \frac{c}{R\Omega}\left(1 - \frac{R\bar{\omega}^2}{2gc}\right). \tag{13}$$

The elements of the transformation matrix, turning the particle–field system to the principal axis, are obtained in terms of the physically meaningful quantities $\Omega_r, \bar{\omega}$ after some rather long but straightforward manipulations [7]. They read

$$t_0^r = \frac{\eta\Omega_r}{\sqrt{\left(\Omega_r^2 - \bar{\omega}^2\right)^2 + \frac{\eta^2}{2}(3\Omega_r^2 - \bar{\omega}^2) + 4g^2\Omega_r^2}}, \tag{14}$$

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r. \tag{15}$$

Let us now consider the eigenstates of our system, $|n_0, n_1, n_2, \ldots\rangle$, represented by the normalized eigenfunctions, written in terms of the normal coordinates $\{Q_r\}$,

$$\phi_{l_0 l_1 l_2 \ldots}(Q, t) = \prod_s \left[\sqrt{\frac{2^{l_s}}{l_s!}} H_{l_s}\left(\sqrt{\frac{\Omega_s}{\hbar}} Q_s\right)\right] \Gamma_0 \, \mathrm{e}^{-\mathrm{i}\sum_s (l_s + \frac{1}{2})\Omega_s t}, \tag{16}$$

where H_{l_s} stands for the l_s th Hermite polynomial and Γ_0 is the normalized vacuum eigenfunction.

We introduce *renormalized* coordinates q_0' and $\{q_i'\}$ for the *dressed* particle and the *dressed* field, respectively, defined by

$$\sqrt{\bar{\omega}_\mu}q_\mu' = \sum_r t_\mu^r \sqrt{\Omega_r} Q_r, \tag{17}$$

where $\bar{\omega}_\mu = \{\bar{\omega}, \omega_i\}$. In terms of the renormalized coordinates, we define for a fixed instant, $t = 0$, *dressed* states, $|\kappa_0, \kappa_1, \kappa_2, \ldots\rangle$ by means of the complete orthonormal set of functions [7]

$$\psi_{\kappa_0\kappa_1\ldots}(q') = \prod_\mu \left[\sqrt{\frac{2^{\kappa_\mu}}{\kappa_\mu!}} H_{\kappa_\mu}\left(\sqrt{\frac{\bar{\omega}_\mu}{\hbar}} q_\mu'\right)\right]\Gamma_0, \tag{18}$$

where $q_\mu' = \{q_0', q_i'\}$. Note that the ground state Γ_0 in the above equation is the same as in equation (16). The invariance of the ground state is due to our definition of renormalized coordinates given by equation (17). Each function $\psi_{\kappa_0\kappa_1\ldots}(q')$ describes a state in which the dressed oscillator q_μ' is in its κ_μ th excited state. In terms of the bare coordinates, the renormalized coordinates are expressed as

$$q_\mu' = \sum_\nu \alpha_{\mu\nu} q_\nu, \tag{19}$$

where

$$\alpha_{\mu\nu} = \frac{1}{\sqrt{\bar{\omega}_\mu}} \sum_r t_\mu^r t_\nu^r \sqrt{\Omega_r}. \tag{20}$$

Remark that the introduction of the renormalized coordinates implies, differently from the bare vacuum, the stability of the dressed vacuum state since, by construction, it is identical to the ground state of the interacting Hamiltonian (3). Also it is important to note that the renormalized coordinates and the dressed states, defined in equations (17) and (18) are *new* collective objects, *different* from the normal coordinates Q and the eigenstates (16). Since transformation (17) is not orthogonal, the Hamiltonian is not diagonal in the renormalized coordinates. Thus, distinctly from the eigenstates, our dressed states are all unstable, except for the ground dressed ($\{\kappa_\mu = 0\}$) state. We shall assume that the dressed states (18) are the physically appropriate states to describe the time evolution of superpositions of states of the mechanical oscillator, taking into account non-perturbatively the effect of the interaction with the environment. This is an alternative to the use of states written in terms of the bare coordinates q_μ, which would require a perturbative renormalization procedure to correct order by order the oscillator frequency.

Let us consider, for instance, the particular dressed state $\left|\Gamma_1^\mu(0)\right\rangle$, represented by the wavefunction $\psi_{00\cdots1(\mu)0\cdots}(q')$. It can be seen as describing, at a given instant, the configuration in which *only* the μth dressed oscillator is in the 'first excited level', all others being in their 'ground states'. These 'levels' should not be confused with the *stationary* states given by equation (16); from now on, we shall use such a terminology to facilitate the reference to dressed states. As shown in [7], the time evolution of the state $\left|\Gamma_1^\mu\right\rangle$ is given by

$$\left|\Gamma_1^\mu(t)\right\rangle = \sum_\nu f_{\mu\nu}(t)\left|\Gamma_1^\nu(0)\right\rangle, \tag{21}$$

where

$$f_{\mu\nu}(t) = \sum_s t_\mu^s t_\nu^s \,\mathrm{e}^{-\mathrm{i}\Omega_s t}. \tag{22}$$

Moreover, it can be shown that for all μ,

$$\sum_{\nu} |f_{\mu\nu}(t)|^2 = 1, \tag{23}$$

which allows us to interpret the coefficients $f_{\mu\nu}(t)$ as probability amplitudes; for example, $f_{00}(t)$ is the probability amplitude that, if the dressed particle is in the first excited state at $t = 0$, it remains excited at time t, while $f_{0i}(t)$ represents the probability amplitude that the ith dressed harmonic mode of the cavity be at the first excited level. Also, the elements of the matrix $T^{l_0 l_1 \cdots}_{\kappa_0 \kappa_1 \cdots}$, connecting the dressed states to the eigenstates can be evaluated. For the dressed state in which only the κ_μ th dressed oscillator is in the Nth excited level, all other being in the ground state, we have [7]

$$T^{l_0 l_1 \cdots}_{0,0,\cdots} = \left(\frac{N!}{l_0! l_1! \cdots}\right)^{\frac{1}{2}} \left(t^0_\mu\right)^{l_0} \left(t^1_\mu\right)^{l_1} \cdots, \tag{24}$$

where $l_0 + l_1 + \cdots = N$.

3. Behavior of the superposition of states in a cavity

We now analyze the situation in which, at time $t = 0$, the environment oscillators are in their ground states and the particle is in a superposition of dressed states, and the whole system is allowed to evolve in time. Specifically, we take, as the initial particle dressed state, an arbitrary superposition of the first excited and the ground states,

$$|\psi\rangle = \sqrt{\xi}\left|\Gamma^0_1(0)\right\rangle + \sqrt{1-\xi}\, \mathrm{e}^{\mathrm{i}\phi} \left|\Gamma_0\right\rangle, \tag{25}$$

where $0 < \xi < 1$. While the ground state is stable, the particle first-excited state evolves in time according to equation (21), that is, $\left|\Gamma^0_1(t)\right\rangle = \sum_\nu f_{0\nu}(t)\left|\Gamma^\nu_1(0)\right\rangle$ with $f_{0\nu}(t) = \sum_s t^s_0 t^s_\nu\, \mathrm{e}^{-\mathrm{i}\Omega_s t}$. At the instant t, the state of the system is given by the density matrix

$$\varrho(t) = \mathrm{e}^{-\mathrm{i}Ht} |\psi\rangle\langle\psi|\, \mathrm{e}^{\mathrm{i}Ht}. \tag{26}$$

We are interested in studying the influence of the environment on the time evolution of the state of the subsystem corresponding to the particle. Thus, we take the trace over all the degrees of freedom associated with the field, that is, we consider the reduced density matrix

$$\rho(t) = \sum_{\{k_i=0\}}^{\infty} \langle k_1, k_2, \ldots |\mathrm{e}^{-\mathrm{i}Ht}|\psi\rangle\langle\psi|\mathrm{e}^{\mathrm{i}Ht}|k_1, k_2, \ldots\rangle, \tag{27}$$

whose elements between dressed particle states are

$$\rho_{mn}(t) = \langle m|\rho|n\rangle = \sum_{\{k_i=0\}}^{\infty} \langle m, k_1, k_2, \ldots |\mathrm{e}^{-\mathrm{i}Ht}|\psi\rangle\langle\psi|\mathrm{e}^{\mathrm{i}Ht}|n, k_1, k_2, \ldots\rangle. \tag{28}$$

Replacing equation (25) in equation (28) we get

$$\begin{aligned} \rho_{mn}(t) = \sum_{\{k_i=0\}}^{\infty} &\big\{\xi \mathcal{A}^{mk_1k_2\cdots}_{100\ldots}(t)\mathcal{A}^{nk_1k_2\ldots *}_{100\ldots}(t) + (1-\xi)\mathcal{A}^{mk_1k_2\cdots}_{000\ldots}(t)\mathcal{A}^{nk_1k_2\ldots *}_{000\ldots}(t) \\ &+ \sqrt{\xi(1-\xi)}\, \mathrm{e}^{-\mathrm{i}\phi} \mathcal{A}^{mk_1k_2\cdots}_{100\ldots}(t)\mathcal{A}^{nk_1k_2\ldots *}_{000\ldots}(t) \\ &+ \sqrt{\xi(1-\xi)}\, \mathrm{e}^{\mathrm{i}\phi} \mathcal{A}^{mk_1k_2\cdots}_{000\ldots}(t)\mathcal{A}^{nk_1k_2\ldots *}_{100\ldots}(t)\big\}, \end{aligned} \tag{29}$$

where

$$\mathcal{A}^{m_0 m_1 m_2 \cdots}_{n_0 n_1 n_2 \ldots}(t) = \langle m_0, m_1, m_2, \ldots |\mathrm{e}^{-\mathrm{i}Ht}|n_0, n_1, n_2, \ldots\rangle \tag{30}$$

are the probability amplitudes between the dressed states. These probability amplitudes can be computed using equations (22) and (24). We then have

$$\mathcal{A}_{000\ldots}^{m_0 m_1 m_2\ldots}(t) = \mathrm{e}^{-\mathrm{i}E_0 t}\delta_{0m_0}\delta_{0m_1}\delta_{0m_2}\ldots, \tag{31}$$

$$\mathcal{A}_{100\ldots}^{m_0 m_1 m_2\ldots}(t) = \mathrm{e}^{-\mathrm{i}E_0 t}\sum_{\mu} f_{0\mu}(t)\delta_{1m_\mu}\prod_{\nu\neq\mu}\delta_{0m_\nu}, \tag{32}$$

where $E_0 = \sum_r \Omega_r/2$ is the ground-state energy of the system. Substituting the above equations into equation (29), we obtain, after summing over the k_i's,

$$\rho_{mn}(t) = \left(1-\xi+\xi\sum_k |f_{0k}(t)|^2\right)\delta_{0m}\delta_{0n} + \xi|f_{00}(t)|^2\delta_{1m}\delta_{1n} + \sqrt{\xi(1-\xi)}\,\mathrm{e}^{-\mathrm{i}\phi}f_{00}(t)\delta_{1m}\delta_{0n} + \sqrt{\xi(1-\xi)}\,\mathrm{e}^{\mathrm{i}\phi}f_{00}^{*}(t)\delta_{0m}\delta_{1n}. \tag{33}$$

Using that $\sum_\mu |f_{0\mu}|^2 = 1$, the nonzero elements of the reduced density matrix are given by

$$\rho_{00}(t) = 1-\xi|f_{00}(t)|^2, \tag{34}$$

$$\rho_{11}(t) = \xi|f_{00}(t)|^2, \tag{35}$$

$$\rho_{10}(t) = \sqrt{\xi(1-\xi)}\,\mathrm{e}^{-\mathrm{i}\phi}f_{00}(t), \tag{36}$$

$$\rho_{01}(t) = \sqrt{\xi(1-\xi)}\,\mathrm{e}^{\mathrm{i}\phi}f_{00}^{*}(t). \tag{37}$$

Note that $\mathrm{Tr}[\rho(t)] = 1$, which means that the reduced density matrix $\rho(t)$ does represent a physical state of the particle; also, it is not a pure state, since $\mathrm{Tr}[\rho^2(t)] \neq 1$. The degree of impurity of a quantum state can be quantified by the departure of the idempotent property; in the present case, we have

$$D(t;\xi) = 1-\mathrm{Tr}[\rho^2] = 2\xi^2|f_{00}(t)|^2(1-|f_{00}(t)|^2). \tag{38}$$

We are thus left with the calculation of the quantity $f_{00}(t)$, which will be done for the particular situations of an arbitrarily large cavity ($R\to\infty$) and of a small cavity.

3.1. Large cavity

For an arbitrarily large value of R, equation (14) reduces to

$$t_0^r \to \lim_{R\to\infty}\frac{\sqrt{4g/\pi}\,\Omega\sqrt{\pi c/R}}{\sqrt{(\Omega^2-\bar{\omega}^2)^2+4g^2\Omega^2}}. \tag{39}$$

Also, in this limit, $\Delta\omega = \pi c/R \to \mathrm{d}\omega = \mathrm{d}\Omega$ and the expression for the quantity $f_{00}(t)$, given by equation (22) can be cast in the form

$$f_{00}(t) = \frac{4g}{\pi}\int_0^\infty \mathrm{d}\Omega\frac{\Omega^2\,\mathrm{e}^{-\mathrm{i}\Omega t}}{(\Omega^2-\bar{\omega}^2)^2+4g^2\Omega^2}. \tag{40}$$

The real part of $f_{00}(t)$ can be evaluated directly using the residue theorem. Defining the parameter κ by

$$\kappa = \sqrt{\bar{\omega}^2-g^2}, \tag{41}$$

we obtain, for the three distinct cases,

(a) $\kappa^2 > 0$:

$$f_{00}(t) = \mathrm{e}^{-gt}\left[\cos\kappa t - \frac{g}{\kappa}\sin\kappa t\right] + \mathrm{i}G(t;\bar{\omega},g); \tag{42}$$

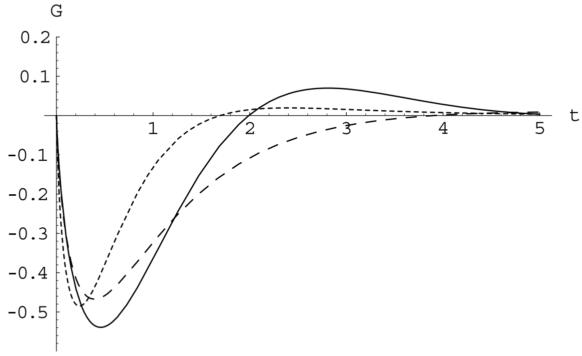

Figure 1. Behavior of $G(t;\bar{\omega},g)$ as a function of t for the three distinct cases: $\bar{\omega}=1.5$ and $g=1.0$ (full line); $\bar{\omega}=1.0$ and $g=1.2$ (dashed line); and $\bar{\omega}=g=2.0$ (dotted line), in arbitrary units.

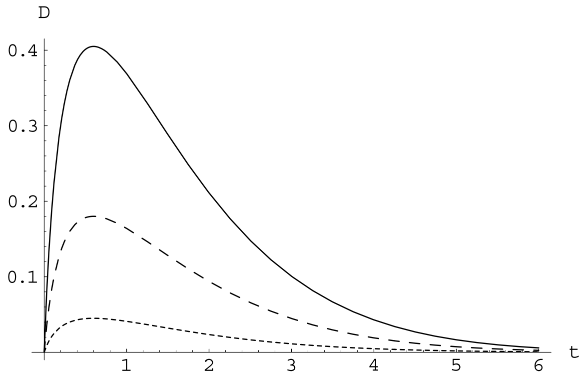

Figure 2. Behavior of $D(t;\xi)$ as a function of t, with $\bar{\omega}=1.0$ and $g=0.5$ fixed (in arbitrary units), for some values of ξ: 0.3, 0.6 and 0.9 (dotted, dashed and full lines, respectively).

(b) $\kappa^2=0$:

$$f_{00}(t)=\mathrm{e}^{-gt}\,[1-gt]+\mathrm{i}G(t;\bar{\omega},g); \tag{43}$$

and (c) $\kappa^2<0$:

$$f_{00}(t)=\mathrm{e}^{-gt}\left[\cosh|\kappa|t-\frac{g}{|\kappa|}\sinh|\kappa|t\right]+\mathrm{i}G(t;\bar{\omega},g), \tag{44}$$

where the function $G(t;\bar{\omega},g)$ is given by

$$G(t;\bar{\omega},g)=-\frac{4g}{\pi}\int_0^\infty \mathrm{d}y\,\frac{y^2\sin yt}{(y^2-\bar{\omega}^2)^2+4g^2y^2}. \tag{45}$$

The overall behavior of the function $G(t;\bar{\omega},g)$ is illustrated in figure 1; we see then that, in all cases, the real and imaginary parts of $f_{00}(t)$ decay with the time, for large times. This aspect dictates the behavior of the degree of purity, as a function of time, as illustrated in figure 2 for a case with $g<\bar{\omega}$; similar results are obtained for the other regimes.

Replacing $f_{00}(t)$, given by (42), (43) or (44) into equations (34)–(37) leads to the elements of the reduced density matrix; simpler expressions are obtained for large times ($t \gg 1/\bar{\omega}$) at both *weak*- and *strong-coupling* regimes. For large t, $G(t; \bar{\omega}, g)$ can be approximated as

$$G(t; \bar{\omega}, g) \approx \frac{8g}{\pi \bar{\omega}^4 t^3} \qquad \left(t \gg \frac{1}{\bar{\omega}}\right). \tag{46}$$

Thus, in the limit case of weak coupling between the particle and the environment, $g \ll \bar{\omega}$ (corresponding to $\kappa \approx \bar{\omega}$), the large-time approximation of the reduced density matrix gives

$$\rho_{11}(t) \approx \xi \left\{ \mathrm{e}^{-2gt} \left[\cos \bar{\omega} t - \frac{g}{\bar{\omega}} \sin \bar{\omega} t \right]^2 + \frac{64 g^2}{\pi^2 \bar{\omega}^8 t^6} \right\}, \tag{47}$$

$$\rho_{00}(t) = 1 - \rho_{11}(t), \tag{48}$$

$$\rho_{10}(t) \approx \sqrt{\xi(1-\xi)}\, \mathrm{e}^{-\mathrm{i}\phi} \left\{ \mathrm{e}^{-gt} \left[\cos \bar{\omega} t - \frac{g}{\bar{\omega}} \sin \bar{\omega} t \right] + \mathrm{i} \frac{8g}{\pi \bar{\omega}^4 t^3} \right\}, \tag{49}$$

$$\rho_{01}(t) = \rho_{10}^{*}(t). \tag{50}$$

Similar results can be obtained for the case of a strong coupling between the particle and the environment, that is, when $g \gg \bar{\omega}$ (i.e. $|\kappa| \approx g$); for large t, we find

$$\rho_{11}(t) \approx \xi \left[\mathrm{e}^{-4gt} + \frac{64 g^2}{\pi^2 \bar{\omega}^8 t^6} \right], \tag{51}$$

$$\rho_{00}(t) = 1 - \rho_{11}(t), \tag{52}$$

$$\rho_{10}(t) \approx \sqrt{\xi(1-\xi)}\, \mathrm{e}^{-\mathrm{i}\phi} \left[\mathrm{e}^{-2gt} + \mathrm{i} \frac{8g}{\pi \bar{\omega}^4 t^3} \right], \tag{53}$$

$$\rho_{01}(t) = \rho_{10}^{*}(t). \tag{54}$$

We see from the above equations that, for both weak- and strong-coupling regimes, in the case of a very large cavity, except for the vacuum–vacuum component which goes to 1, all other matrix elements of the reduced density matrix tend to zero as $t \to \infty$. This means that the particle mixed state $\rho(t)$ tends to the dressed-particle vacuum state as time evolves; thus, for large cavities, the particle system is dissipative. This is also illustrated in figure 2, with $D(t; \xi)$ vanishing as $t \to \infty$ for all values of ξ. Note that the real and imaginary parts of the matrix elements $\rho_{10}(t)$ and $\rho_{01}(t)$, which correspond to interference terms, decay more slowly than the element $\rho_{11}(t)$.

3.2. Small cavity

For a finite cavity, the spectrum of eigenfrequencies is discrete, $\Delta\omega$ is large, and so the approximation made in the case of large cavity does not apply; no analytical result can be obtained for $f_{00}(t)$, given by equation (22), in this case. However, if the cavity is sufficiently small, the frequencies Ω_r can be determined as follows [8]. In terms of the small dimensionless parameter

$$\delta = \frac{g}{\Delta\omega} = \frac{gR}{\pi c}, \tag{55}$$

equation (13) is rewritten as

$$\cot(\pi\theta) = \frac{\theta}{2\delta} + \frac{1}{\pi\theta}\left(1 - \frac{\pi \bar{\omega}^2 \delta}{2g^2}\right), \tag{56}$$

where $\theta = \Omega/\Delta\omega$. With $\delta \ll 1$, which corresponds to $R \ll \pi c/g$ (a small cavity), as plots of both sides of equation (56) as functions of θ are drawn, one sees that for $k = 1, 2, \ldots$, the solutions of equation (56) can be written as $\theta_k \approx k + \epsilon_k$ with $\epsilon_k \ll 1$; then, expanding $\cot(\pi\epsilon_k)$ for small ϵ_k, one finds

$$\Omega_k \approx \Delta\omega\left(k + \frac{2\delta}{\pi k}\right) = \frac{g}{\delta}\left(k + \frac{2\delta}{\pi k}\right). \tag{57}$$

If we further impose that $\delta < 2g^2/\pi\bar{\omega}^2$, a condition compatible with $\delta \ll 1$, then Ω_0 is found to be very close to $\bar{\omega}$, that is,

$$\Omega_0 \approx \bar{\omega}\left(1 - \frac{\pi\delta}{2}\right). \tag{58}$$

To determine $f_{00}(t)$, we have to calculate the square of the matrix elements $\left(t_0^0\right)^2$ and $\left(t_k^0\right)^2$. From equation (15), using that $\omega_k = k\Delta\omega = kg/\delta$, we find (to first order in δ)

$$\left(t_k^0\right)^2 \approx \frac{4\delta}{\pi k^2}\left(t_0^0\right)^2. \tag{59}$$

Now, considering the orthonormalization condition (4), we determine $\left(t_0^0\right)^2$ as

$$\left(t_0^0\right)^2 \approx \left(1 + \frac{4\delta}{\pi}\sum_{k=1}^{\infty} k^{-2}\right)^{-1} = \left(1 + \frac{2}{3}\pi\delta\right)^{-1}, \tag{60}$$

where we have used that $\zeta(2) = \sum_{k=1}^{\infty} k^{-2} = \pi^2/6$. We thus obtain, for sufficiently small cavities ($\delta \ll 1$),

$$f_{00}(t) \approx \left(1 + \frac{2}{3}\pi\delta\right)^{-1}\left[\mathrm{e}^{-\mathrm{i}\bar{\omega}(1-\frac{\pi\delta}{2})t} + \sum_{k=1}^{\infty}\frac{4\delta}{\pi k^2}\,\mathrm{e}^{-\mathrm{i}\frac{g}{\delta}(k+\frac{2\delta}{\pi k})t}\right]. \tag{61}$$

Note that, if we had calculated $\left(t_0^0\right)^2$ by approximating directly equation (14) for small δ, instead of using the normalization condition (4), $f_{00}(t)$ would not satisfy the requirement $|f_{00}(t)|^2 \leqslant 1$ with the equality occurring when $t = 0$. Also, it is worth mentioning that the approximation we made holds independently of the strength of the coupling between the particle and the environment and, thus, it applies for both weak and strong limits.

Substituting $f_{00}(t)$, given by equation (61), into equations (34)–(37) leads to the elements of the reduced density matrix for the case of a small cavity. Distinctly from the case of a very large cavity, now, these matrix elements do not tend to zero as time evolves; in fact, they oscillate, never reaching zero. In particular, we find that

$$\begin{aligned}\rho_{11}(t) \approx \xi\left(1 + \frac{2}{3}\pi\delta\right)^{-2}&\left\{1 + \frac{8\delta}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos\left[\bar{\omega}\left(1 - \frac{\pi\delta}{2}\right) - \frac{g}{\delta}\left(k + \frac{2\delta}{\pi k}\right)\right]t\right.\\ &\left. + \frac{16\delta^2}{\pi^2}\sum_{k,l=1}^{\infty}\frac{1}{k^2l^2}\cos\left[\left(\frac{g}{\delta} - \frac{2g}{\pi kl}\right)(k-l)\right]t\right\}.\end{aligned} \tag{62}$$

Note that $\rho_{11}(t)$ is an oscillating, nonperiodic function of time. A lower bound for ρ_{11} can be found if one replaces the cosine terms in equation (62) by -1; in fact, to order δ^2, one finds

$$\rho_{11}(t) > \xi\left[1 - \frac{8}{3}\pi\delta + \frac{8}{9}\pi^2\delta^2\right]. \tag{63}$$

This means that the particle state never reduces to the dressed-particle vacuum state; the particle system, maintaining itself in a mixed state, keeps exchanging quanta with the environment.

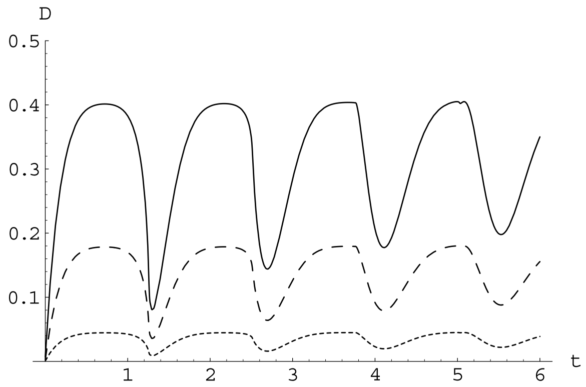

Figure 3. Behavior of $D(t;\xi)$ as a function of t for $\delta = 0.1$, with $\bar{\omega} = 1.0$ and $g = 0.5$ fixed (in arbitrary units), for some values of ξ: 0.3, 0.6 and 0.9 (dotted, dashed and full lines, respectively).

The degree of impurity of this particle state at time t can be calculated using equation (38) or, equivalently, with

$$D(t;\xi) = 2\rho_{11}(t)\,[\xi - \rho_{11}(t)]\,. \tag{64}$$

This quantity is plotted in figure 3, for the same values of $\bar{\omega}$, g and ξ as those used in figure 2 (for comparison), fixing a specific value of δ; one clearly sees the oscillatory, nonperiodic, behavior mentioned above. Therefore, in the case of a small cavity, for both weak- and strong-coupling regimes, the particle system is not dissipative; it keeps exchanging energy with the environment. For example, in the case of weak coupling, a physically interesting situation occurs when interactions of electromagnetic type are involved. In this case, we can take $g = \alpha\bar{\omega}$, where α is the fine-structure constant, $\alpha \sim 1/137$. For a frequency $\bar{\omega} \sim 2 \times 10^{11}\ \mathrm{s}^{-1}$ (in the microwave band) and a cavity with $R \sim 10^{-2}$ m, we have $\delta \sim 0.016$ and, so, the diagonal element $\rho_{11}(t)$ will never fall below the value $\sim 0.87\xi$. For strong coupling, $g = \beta\bar{\omega}$ $(\beta > 1)$, a similar result holds for a smaller cavity. The same kind of oscillatory behavior can be verified for the nondiagonal elements of the density matrix. In other words, the system confined in a small cavity is not dissipative, contrary to the case of a large cavity.

4. Concluding remarks

We have presented in this paper a nonperturbative treatment of a quantum system consisting of a particle, in the harmonic approximation, coupled to an environment modeled by noninteracting oscillators. We have used renormalized coordinates in terms of which dressed states can be constructed. These states allow us to naturally separate the system into the dressed (physically observed) particle and the dressed environment by means of the conveniently chosen renormalized coordinates, q_0' and q_j', associated respectively with the dressed particle and with the dressed oscillators composing the environment. The dressed particle will contain automatically all the effects of the environment in it. Using this formalism we perform a nonperturbative study of the time evolution of a superposition of the ground state and the first excited state of the particle, the system being confined in a perfectly reflecting cavity of radius R.

In the $R \to \infty$ limit, we find dissipation (damping) with dominance of the interference terms of the density matrix, in both weak- and strong-coupling regimes. For small values of

R, the system is not dissipative, presenting stability. In both cases we have obtained results in agreement with expected behaviors. However, we do not simply recover well-known features with our method. For instance, as far as we know, the corrections to the exponential decay in equations (47)–(50) and from equations (51) to (54) have not been reported before in the literature. Also, the absence of dissipation in a small cavity can be understood as the analogue of spontaneous emission inhibition for an atom, which has a qualitative explanation when the lowest resonance cavity frequency is larger than the oscillator frequency. But also in this case we are able, with our method, to give rigorous expressions for the elements of the density matrix (in an analogous manner as has been done in [8] for the size of cavities ensuring stability of excited atoms).

It is worth stressing that the renormalized coordinates and the dressed states, defined in equations (17) and (18), allow an exact treatment of the problem, in which one completely avoids the use of perturbation theory. As already noted in the text below equation (20), the renormalized coordinates and dressed states are *new* collective objects, *different* from the normal coordinates Q and the eigenstates (16). Distinctly from the eigenstates, our dressed states are all unstable, except for the ground dressed ($\{\kappa_\mu = 0\}$) state. We assume that the dressed states (18) are the physically meaningful states, instead of the ones written in terms of the bare coordinates q_μ. This can be seen as analogous to the wavefunction renormalization in quantum field theory, which justifies referring to q'_μ as *renormalized* coordinates. Also, it is worth mentioning that the introduction of renormalized coordinates naturally ensures that the dressed vacuum state is stable, contrarily to the bare perturbative vacuum, since, by construction, it is identical to the ground state of the interacting Hamiltonian (3). Remember that the invariance of the ground state is due to our definition of renormalized coordinates given by equation (17).

As already explained in the introduction, our formalism and results are restricted to a linear model, but it has the advantage of being nonperturbatively solvable. More realistic physical situations, however, would require a nonlinear coupling between the particle and the environment. It is nevertheless envisageable that such a study of complex nonlinear systems could be handled with our nonperturbative method. Indeed, the concept of renormalized coordinates has already been extended to a nonlinear model in [12]. Also, the generalization of the work presented in the present paper to the case of finite temperature, as well as the study of dressed coherent states, is in progress and will be presented elsewhere.

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